or

$$\Phi^T \Delta M \Phi = I - m_A \quad (1)$$

where m_A is the nondiagonal $\Phi^T M_A \Phi$ having unit diagonal elements. Since Φ is rectangular (and has no inverse) there are an infinite number of ΔM matrices which will satisfy Eq. (1). It is possible to find that ΔM which has some minimum weighted Euclidean norm within the constraint of Eq. (1).

It is physically reasonable and mathematically convenient to minimize the function

$$\epsilon = \| N^{-1} \Delta M N^{-1} \| \quad (2)$$

where $N = M_A^{1/2}$ as in Ref. 6. Note that it is not necessary to compute N since only $N^2 = M_A$ appears in the final result.

Defining a Lagrangian multiplier λ_{ij} for each element in Eq. (1), the following Lagrangian function may be written:

$$\psi = \epsilon + \sum_{i=1}^{m} \sum_{j=1}^{m} \lambda_{ij} (\Phi^T \Delta M \Phi - I + m_A)_{ij} \quad (3)$$

Differentiating Eq. (3) with respect to each element of ΔM and setting these results equal to zero will satisfy the minimization of Eq. (2) if the constraint of Eq. (1) is also satisfied. This process results in the matrix equation

$$2M_A^{-1} \Delta M M_A^{-1} + \Phi \Lambda^T \Phi^T = 0$$

or

$$\Delta M = -\tfrac{1}{2} M_A \Phi \Lambda^T \Phi^T M_A \quad (4)$$

where Λ is a square $(m \times m)$ matrix of λ_{ij}.
Substituting Eq. (4) into Eq. (1) allows the solution for Λ

$$\Lambda = -2m_A^{-1} (I - m_A) m_A^{-1} \quad (5)$$

which is then substituted into Eq. (4) to obtain

$$\Delta M = M_A \Phi m_A^{-1} (I - m_A) m_A^{-1} \Phi^T M_A \quad (6)$$

Comments

Equation (6) is an easily evaluated expression for the incremental changes in the mass matrix to make it consistent with the measured modes. Note that ΔM is symmetrical as is theoretically necessary. If some minimization other than that of Eq. (2) were desired, the same process would result in a similar but probably less appealing expression than Eq. (6). If other information were available which indicated that different values of the generalized masses, $\phi_i^T (M_A + \Delta M) \Phi_i$, were more meaningful, this information would result in a diagonal matrix other than the unit matrix of Eq. (1) and the resulting ΔM would yield these values.

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Finite-Element Solution of the Supersonic Flutter of Conical Shells

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Nomenclature

$[a_I]$, $[A_I]$	= element and system aerodynamic damping matrix
$[a_R]$, $[A_R]$	= element and system aerodynamic stiffness matrix
$[k]$, $[K]$	= element and system stiffness matrix
$[m]$, $[M]$	= element and system mass matrix
M	= local Mach number
n	= number of circumferential half-waves
p	= pressure
$\{q\}$	= displacement vector $(uvw)^T$
$\{\bar{q}\}$, $\{Q\}$	= element and system nodal degrees-of-freedom vector
r	= radius
s	= meridional coordinate
S	= total shell meridional length
t	= time
u	= meridional displacement
v	= tangential displacement
V	= local air velocity
w	= radial displacement
β	= rotational nodal degree of freedom
δ	= variational operator
θ	= circumferential coordinate
κ	= reduced frequency, $\omega S/V$
λ	= dynamic pressure parameter, $\rho_a V^2 R_1^3 / D(M^2-1)^{1/2}$
ρ_a	= air mass density
ω	= frequency, $\omega_R + i\omega_I$, $i = (-1)^{1/2}$

Subscripts

a	= air
cr	= critical

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I = imaginary
R = real
∞ = freestream value

Superscript
T = transposed matrix

I. Introduction

RECENTLY, extensive analytical and experimental research has been performed in the field of aeroelasticity of plates and shells. Much of the work has been devoted to the problem of flat plates, curved plates, and circular cylindrical shells.[1,2] The problems of conical shells and general shells of revolution, despite their importance in high-speed vehicles, have received little attention. To the authors' knowledge, the only works dealing with conical shells have been published by Shulman,[3] Dzygadlo,[4] Dixon and Hudson,[5,6] and Librescu[2] using a quasisteady aerodynamic theory and the case of general shells of revolution by Bismarck-Nasr[7] where a full linearized potential aerodynamic theory has been used. The finite-element method has been applied to supersonic panel flutter by several investigators.[8-12] In the following an attempt is made to apply this method to the case of conical shells. Numerical results are presented and are compared with previous analytical solutions.

II. Theoretical Formulation

Hamiton's principle for the problem at hand can be expressed as

$$\int_{t_0}^{t_1} \delta(T-U)\,\mathrm{d}t + \int_{t_0}^{t_1} \delta W \mathrm{d}t = 0 \tag{1}$$

In the following analysis, the in-plane inertias are retained in the kinetic energy expression T, and rotary inertias are neglected. The strain energy due to small deformation U is expressed using Novozhilov's theory of thin shells.[13] We notice that in previous analytical formulations,[2,3,5,6] the Donnell-Mushtari theory of thin shells[13] was used; thus, the tangential terms in the change of curvature and twist expressions were neglected. It is known that such simplifications in vibration problems, depending on the shell geometry and/or the number of circumferential nodes, can lead to great errors in the frequency calculations.[13] The work done by the external aerodynamic forces W, using a first-order, high Mach number approximation to the linear potential flow theory with the inclusion of a correction term for curvature,[1,2] reads,

$$W = -\int_0^{2\pi}\int_{s_1}^{s_2} \frac{\rho_a V^2}{\sqrt{M^2-1}}\left[\frac{\partial w}{\partial s} + \frac{1}{V}\frac{M^2-1}{M^2-2}\frac{\partial w}{\partial t} - \frac{w}{2r\sqrt{M^2-1}}\right] wr\mathrm{d}\theta\mathrm{d}s \tag{2}$$

For the present finite-element formulation, a conical frustum element will be used. For the field variable expressions, we write

$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \begin{bmatrix} \cos n\theta & & \\ & \sin n\theta & \\ & & \cos n\theta \end{bmatrix} \begin{Bmatrix} u(s) \\ v(s) \\ w(s) \end{Bmatrix} \tag{3}$$

The orthogonality properties of the triangular functions used here resulted in uncoupled equations of motion for the different harmonics. Zero-order Hermitian interpolation functions will be used to approximate the in-plane values $u(s)$ and $v(s)$, and first-order Hermitian interpolation functions will be used to approximate the radial value $w(s)$, in order to express them as functions of the nodal degrees of freedom $(uvw\beta)^T$. We notice that the interpolation functions used here preserve $C^{(0)}$ continuity for u and v, and $C^{(1)}$ continuity for w, thus satisfying the compatibility and completeness requirements of the finite-element method.[14] Substituting the approximate displacement expressions into the Hamilton principle, Eq. (1), minimizing with respect to the nodal degrees of freedom, and assuming an exponential motion in the form

$$\{\bar{q}\} = e^{i\omega t}\{\bar{q}_0\} \tag{4}$$

we obtain for each element the following matrix equation

$$[k]\{\bar{q}_0\} - \omega^2[m]\{\bar{q}_0\} - V^2[[a_R] + i\kappa[a_I]]\{\bar{q}_0\} = \{0\} \tag{5}$$

The element stiffness matrix was obtained using numerical integrations, while the mass and aerodynamic matrices were integrated analytically. Now, using the standard assembly technique of the finite-element method and applying the appropriate boundary conditions, we get the following system equations for the whole structure,

$$[[K] - \omega^2[M] - V^2[[A_R] + i\kappa[A_I]]]\{Q_0\} = \{0\} \tag{6}$$

This presents a complex eigenvalue problem with ω regarded as the eigenvalue. The borderline of aeroelastic stability will be determined when the imaginary part of one of the roots changes sign from positive to negative. The stability boundaries were obtained by fixing the number of circumferential half wavelength n, while varying the reduced frequency κ until an instability occurs. The process was then repeated for different values of n until the minimum critical dynamic pressure was found.

III. Numerical Results

In this section, some results of the calculations performed are reported and compared with the results of other investigators. The first problem presented is a free vibration analysis of a shell of revolution with negative Gaussian curvature, Fig. 1. The results obtained are shown in Table 1 and compared with those of Adelman et al.,[15] who presented a finite-element solution for the problem using a curved axisymmetric shell element. The results show that good agreement was obtained using only ten elements. The second example treated is an aeroelastic analysis of an open cone. In the numerical calculations the following parameters were

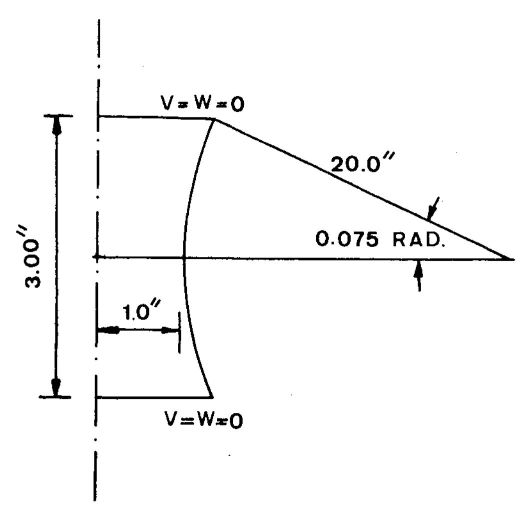

Fig. 1 Shell of revolution with a negative Gaussian curvature, $E=0.91$ lb/in.^{2}, $\nu=0.3$, $\rho=1$ lb s^2/in.^{4}, $h=0.001$ in.

Table 1 Nondimensional frequency $\Omega = \omega R[(\rho/E)(1-\nu^2)]^{1/2}$ of a shell of revolution with negative Gaussian curvature. In calculating Ω, R was taken as = 1, as was done in Ref. 15.

		Present analysis	
n	Ref. 15	10 elements	20 elements
1	0.368	0.3701	0.3685
2	0.157	0.1589	0.1575
3	0.0628	0.0641	0.0632
4	0.0197	0.02053	0.01985
5	0.00779	0.008577	0.00784
6	0.01923	0.02035	0.01945
7	0.02804	0.02988	0.02847
8	0.02580	0.03013	0.02659
9	0.0240	0.02731	0.02432
10	0.0292	0.03203	0.02941

Table 2 Dynamic pressure parameter λ_{cr} for the conical shell

Source	λ_{cr}	n_{cr}
Shulman,[3] Galerkin: 4 terms	669	6
Librescu,[2] Galerkin: 2 terms	448	5
Dixon and Hudson,[5]		
Galerkin: 4 terms	492	5
Galerkin: 8 terms	588	5
Galerkin: 12 terms	590	5
Present analysis, 10 elements		
A_1	670	6
A_2	662	6
A_3	702	6

used: Young's modulus, $E=6.5\times10^6$ lb/in.^{2}, Poisson's ratio, $\nu=0.29$, material mass density, $\rho=8.33\times10^{-4}$ lb s^2/in.^{4}, shell thickness, $h=0.051$ in., cone semivertex angle, $\phi=5$ deg, $M_\infty=3$, $T_\infty=288.15$ K, $p_{a\infty}=14.696$ lb/in.^{2}, $R_1/h=148$, and $S/R_1=8.13$, where R_1 is the cone small end radius. The boundary conditions assumed were $u=v=w=0$ at both ends. These parameters and boundary conditions have been used in the numerical calculations in order to compare the present results using the finite-element method with the analytical solutions of other investigators,[2,3,5] where the same assumptions were made. The results obtained are shown in Table 2. All the calculations of the present analysis were made using a model of ten elements. In the present analysis, three calculations were made. In the first one, labeled A_1 in Table 2, aerodynamic damping and curvature effects are neglected in the aerodynamic matrix. This reduced to the quasistatic aerodynamic formulation used by other investigators. In the second calculation, labeled A_2 in the table, the curvature effect was included in order to study its influence on the stability boundary. In the third analysis, labeled A_3, the curvature effect and aerodynamic damping were considered. For the shell treated here, we can see that the curvature term in the aerodynamic pressure expression has a small effect on the stability boundary. The aerodynamic damping has a greater effect on the stability boundary and is stabilizing. In the present analysis, Novozhilov's thin shell theory has been used and in-plane inertias were retained. All previous analytical solutions[2,3,5] use the Donnell-Mushtari simplified shell theory and the Galerkin method. However, they differ in the method of application of the Galerkin method and the satisfaction of the boundary conditions. This explains the small disagreement in the results of Table 2.

IV. Conclusions

A finite-element solution of the supersonic flutter of conical shells has been presented. An extension of the present analysis would be the inclusion of internal pressure, axial prestress, and a structural nonlinear formulation of the problem.

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